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# Lefschetz numbers for continuous maps, and periods for expanding maps on infra-nilmanifolds

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**Abstract**

We prove that (1) The Lefschetz number of any continuous map $f$ on an infra-nilmanifold with holonomy group $\Psi$ is $L(f) = \frac{1}{|\Psi|}\sum_{A\in\Psi}\frac{\det(A_*-f_*)}{\det A_*}$; (2) The sets of periods for expanding maps on $n$-dimensional infra-nilmanifolds are uniformly cofinite, i.e., there is a positive integer $m_0$, which depends only on $n$, such that for any integer $m \geq m_0$, for any $n$-dimensional infra-nilmanifold $M$ and for any expanding map $f$ on $M$, there exists a periodic point of $f$ whose least period is exactly $m$. This is a generalization of the main result of [R. Tauraso, Sets of periods for expanding maps on flat manifolds, Monatsh. Math. 128 (1999) 151–157] on flat manifolds to infra-nilmanifolds.


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## 1. Introduction

Let $M$ be a closed manifold and let $f : M \to M$ be a continuous map. A point $x \in M$ is called a *fixed point* of $f$ if $f(x) = x$. It is a *periodic point* (of period $m$) if $f^m(x) = x$ (and $m$ is the smallest such positive integer). This paper is concerned with the fixed points and periodic points of maps. The two numbers $L(f)$ and $N(f)$ are associated with a map $f$, and they are

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most important in the fixed point theory. While $L(f)$ is defined easily using homology, $N(f)$ is quite complicated, and hard to calculate (see below for exact definitions).

A celebrated theorem of Lefschetz states that, *if* $L(f) \neq 0$, *then* $\mathrm{Fix}(f) \neq \emptyset$. Even though $N(f)$ gives better information (i.e., the number of essential fixed point classes), it is hard to calculate. Therefore, there have been attempts to find some relations between these two numbers. In [2], Brooks, Brown, Pak and Taylor show that for a continuous map $f$ on a torus, $|L(f)| = N(f)$. Anosov [1] extended this to nilmanifolds. However, such an equality does not hold on infra-nilmanifolds (see below) as shown in [1]: there is a continuous map $f$ on the Klein bottle for which $N(f) \neq |L(f)|$. If $M$ is an infra-nilmanifold, and $f$ is homotopically periodic or more generally virtually unipotent, then it is known in [8,12] that $L(f) = N(f)$.

For $f : M \to M$, a point $x \in M$ is called a *periodic point of $f$ of period $m$* if $x \in \mathrm{Fix}(f^m)$ but $x \notin \mathrm{Fix}(f^i)$ for any $i < m$. There have been attempts to understand which integers $m$ show up as periodic points of $f$.

Jiang and Llibre show in [6] that there is a positive integer $m_0$ such that for any integer $m \geq m_0$ and for any expanding map $f$ of the $d$-torus $T^d$, there exists a periodic point of $f$ whose least period is exactly $m$. In [16], Tauraso extends this result to each flat manifold. Since any closed manifold which admits an expanding map is homeomorphic to an infra-nilmanifold [4], it is not surprising that all of our results are related to infra-nilmanifolds.

The purpose of the first half of this work is to offer algebraic and practical computation formulas for the Nielsen and Lefschetz numbers of any continuous maps on infra-nilmanifolds in terms of the holonomies of the manifolds. Thereby the deviation from the equality $N(f) = |L(f)|$ can be measured by the holonomy of the manifold.

In the second half of this work we prove that the above "uniform cofiniteness property" of sets of periods for expanding maps is true not only for flat manifolds but also infra-nilmanifolds. Since only infra-nilmanifolds can support expanding maps, our result is the strongest possible generalization.

## 2. Statements of main results

For a continuous map $f : M \to M$, the *Lefschetz number* $L(f)$ of $f$ is defined by

$$L(f) := \sum_k (-1)^k \mathrm{trace}\{(f_*)_k : H_k(M; \mathbb{Q}) \to H_k(M; \mathbb{Q})\}.$$

To define the *Nielsen number* $N(f)$ of $f$, we first define an equivalence relation on $\mathrm{Fix}(f)$ as follows: For $x_0, x_1 \in \mathrm{Fix}(f)$, $x_0 \sim x_1$ if and only if there exists a path $c$ from $x_0$ to $x_1$ such that $c$ is homotopic to $f \circ c$ relative to the end points. An equivalence class of this relation is called a *fixed point class* (=FPC) of $f$. To each FPC $F$, one can assign an integer $\mathrm{ind}(f, F)$. A FPC $F$ is called *essential* if $\mathrm{ind}(f, F) \neq 0$. Now,

$$N(f) := \text{the number of essential fixed point classes.}$$

These two numbers give information on the existence of fixed point sets. If $L(f) \neq 0$, every self-map $g$ of $M$ homotopic to $f$ has a non-empty fixed point set. The Nielsen number is a lower bound for the number of components of the fixed point set of all maps homotopic to $f$. Even though $N(f)$ gives more information than $L(f)$ does, it is harder to calculate.

Let $G$ be a connected and simply connected nilpotent Lie group. Let $\pi$ be a torsion-free, discrete, cocompact subgroup of $G \rtimes C \subset \mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$, where $C$ is a maximal compact subgroup of $\mathrm{Aut}(G)$. Then such a group $\pi$ is called *an almost Bieberbach group* of $G$

and the closed manifold $\pi \setminus G$ is called *an infra-nilmanifold*. It is known that these are exactly the class of *almost flat* Riemannian manifolds.

Let $M = \pi \setminus G$ be an infra-nilmanifold. Let $\Gamma = \pi \cap G$. Then it is L. Auslander's result that (see, for example, [10]) $\Gamma$ is a lattice of $G$, and is the unique maximal normal nilpotent subgroup of $\pi$. The group $\Psi = \pi/\Gamma$ is the holonomy group of $\pi$ or $M$. It sits naturally in $\mathrm{Aut}(G)$.

Any continuous map $f : M \to M$ induces a homomorphism $\pi \to \pi$, which restricts to a homomorphism of a lattice $\Lambda$ of $G$ into itself. This in turn, induces a unique Lie group endomorphism $G \to G$, and hence a Lie algebra endomorphism $f_* : \mathfrak{G} \to \mathfrak{G}$, see below for details. The first main results are the following:

**Theorem 3.4.** *Let $f : M \to M$ be any continuous map on an infra-nilmanifold $M$ with the holonomy group $\Psi$. Then*

$$L(f) = \frac{1}{|\Psi|} \sum_{A \in \Psi} \frac{\det(A_* - f_*)}{\det A_*},$$

$$N(f) = \frac{1}{|\Psi|} \sum_{A \in \Psi} |\det(A_* - f_*)|.$$

*In particular,* $|L(f)| \le N(f)$.

The set of fixed points of $f^k$ is denoted by $\mathrm{Fix}(f^k)$, and $k$ is called a *period* of a *periodic point* $x \in \mathrm{Fix}(f^k)$ of $f$. The set of periods of $f : M \to M$ is defined [16] as

$$\mathcal{P}(f) = \left\{ m \in \mathbb{N} \ \middle|\ \mathrm{Fix}(f^m) \setminus \bigcup_{i=1}^{m-1} \mathrm{Fix}(f^i) \neq \emptyset \right\}.$$

The second main result shows that for a continuous map $f$ on an infra-nilmanifold the set $\mathcal{P}(f)$ contains almost all positive integers, a set of the form $[m_0, \infty)$. More precisely,

**Theorem 4.6.** *Let $n$ be a positive integer. Then there exists a positive integer $m_0$, which depends only on $n$, such that for any integer $m \ge m_0$, for any $n$-dimensional infra-nilmanifold $M$, and for any expanding map $f$ on $M$, there exists a periodic point of $f$ whose least period is exactly $m$.*

## 3. The Lefschetz numbers of maps on infra-nilmanifolds

The affine group $\mathrm{Aff}(G)$ is contained in a bigger semigroup $\mathrm{aff}(G) = G \rtimes \mathrm{Endo}(G)$: Consider the semigroup $\mathrm{Endo}(G)$, the set of all endomorphisms of $G$, under the composition as operation. We form the semi-direct product $G \rtimes \mathrm{Endo}(G)$ and call it $\mathrm{aff}(G)$. With the binary operation

$$(a, A)(b, B) = (a \cdot Ab, AB),$$

the set $\mathrm{aff}(G)$ forms a semigroup with identity $(e, 1)$, where $e \in G$ and $1 \in \mathrm{Endo}(G)$ are the identity elements. The semigroup $\mathrm{aff}(G)$ "acts" on $G$ by

$$(a, A) \cdot x = a \cdot Ax.$$

Note that $(a, A)$ is not a homeomorphism unless $A \in \mathrm{Aut}(G)$. Clearly, $\mathrm{aff}(G)$ is a sub-semigroup of the semigroup of all continuous maps of $G$ into itself, for $((a, A)(b, B))x = (a, A)((b, B)x)$ for all $x \in G$. An element of $\mathrm{aff}(G)$ is called an *affine endomorphism*.

Let $M = \pi \setminus G$ be an infra-nilmanifold. Let $\Gamma = \pi \cap G$, and $\Psi = \pi/\Gamma$ the holonomy group of $M$.

**Lemma 3.1.** *There exists a fully invariant subgroup $\Lambda \subset \Gamma$ of $\pi$ which is of finite index. Therefore, any homomorphism $\theta : \pi \to \pi$ maps $\Lambda$ into itself.*

**Proof.** Suppose the order of the holonomy group is $k$. Let $\Lambda$ be the subgroup of $\pi$ generated by the set

$$\{x^k : x \in \pi\}.$$

Clearly, the generating set is a subset of $\Gamma$ so that our group $\Lambda$ is a subgroup of $\Gamma$. Obviously, any homomorphism $\theta$ sends the generating set $\{x^k : x \in \pi\}$ to itself.

We claim that $\Lambda$ has finite index in $\Gamma$ (and hence in $\pi$). Consider the subgroup $\Gamma(k)$ generated by the set $\{x^k : x \in \Gamma\}$. Since $\Gamma$ is a lattice, it is a polycyclic group. Then $\Gamma(k)$ has finite index in $\Gamma$, see [14, Lemma 4.4]. Since $\Gamma(k) \subset \Lambda$, we find $\Lambda$ has finite index in $\Gamma$. $\square$

Now we explain how a continuous map $f : M \to M$ induces an endomorphism of the Lie algebra, $f_* : \mathfrak{G} \to \mathfrak{G}$, naturally. The continuous map $f : M \to M$ induces a homomorphism $f_\# : \pi \to \pi$. Let $\Lambda$ be a fully invariant subgroup of $\pi$ in Lemma 3.1. Note that $\Lambda$ is a lattice of $G$. Then, the induced homomorphism $f_\# : \pi \to \pi$ restricts to a homomorphism $f_\# : \Lambda \to \Lambda$, which extends to an endomorphism of the Lie group $G$ in a unique way. See [11]. The differential of this map is an endomorphism of the Lie algebra, $f_* : \mathfrak{G} \to \mathfrak{G}$. Therefore, we shall use $f_*$ even for the homomorphism $f_\# : \pi \to \pi$.

**Lemma 3.2.** *For any $g \in G$ and $F \in \mathrm{Endo}(G)$*

$$\det(\mathrm{Ad}(g) - F_*) = \det(I - F_*),$$

*where $F_*$ is the linear transformation of the vector space $\mathfrak{G}$.*

**Proof.** For $g \in G$, $\mu(g)$ denotes the conjugation by $g$ so that $\mu(g)(x) = gxg^{-1}$. Let $\hat{G} = [G, G]$, $\bar{G} = G/[G, G]$. Consider the exact sequence $1 \to \hat{G} \to G \to \bar{G} \to 1$. Then the endomorphisms $F$ and $\mu(g)$ on $G$ induce endomorphisms $\hat{F}$ and $\hat{\mu}(g)$ on $\hat{G}$ and hence, induce endomorphisms $\bar{F}$ and $\bar{\mu}(g)$ on $\bar{G}$. Note that $\bar{\mu}(g) = \mathrm{id}_{\bar{G}}$. Thus the left hand side diagram commutes, which implies that the right hand side diagram commutes:

$$\begin{array}{ccccc} \hat{G} & \longrightarrow & G & \longrightarrow & \bar{G} \\ {\scriptstyle \hat{F}}\downarrow{\scriptstyle \hat{\mu}(g)} & & {\scriptstyle F}\downarrow{\scriptstyle \mu(g)} & & {\scriptstyle \bar{F}}\downarrow{\scriptstyle \mathrm{id}} \\ \hat{G} & \longrightarrow & G & \longrightarrow & \bar{G} \end{array} \qquad \begin{array}{ccccc} \hat{\mathfrak{G}} & \longrightarrow & \mathfrak{G} & \longrightarrow & \bar{\mathfrak{G}} \\ {\scriptstyle \hat{F}_*}\downarrow{\scriptstyle \hat{\mathrm{Ad}}(g)} & & {\scriptstyle F_*}\downarrow{\scriptstyle \mathrm{Ad}(g)} & & {\scriptstyle \bar{F}_*}\downarrow{\scriptstyle \mathrm{id}} \\ \hat{\mathfrak{G}} & \longrightarrow & \mathfrak{G} & \longrightarrow & \bar{\mathfrak{G}} \end{array}$$

Thus we can find a basis of $\mathfrak{G}$ so that $F_*$ and $\mathrm{Ad}(g)$ are of the form

$$F_* = \begin{bmatrix} \hat{F}_* & * \\ 0 & \bar{F}_* \end{bmatrix}, \qquad \mathrm{Ad}(g) = \begin{bmatrix} \hat{\mathrm{Ad}}(g) & * \\ 0 & I \end{bmatrix}.$$

Now the induction on the nilpotency of $G$ implies that there is a basis of $\mathfrak{G}$ such that $F_*$ and $\mathrm{Ad}(g)$ are of the form

$$F_* = \begin{bmatrix} F_{1_*} & & * \\ & \ddots & \\ 0 & & F_{\ell_*} \end{bmatrix}, \qquad \mathrm{Ad}(g) = \begin{bmatrix} I & & * \\ & \ddots & \\ 0 & & I \end{bmatrix}.$$

Hence $\det(\mathrm{Ad}(g) - F_*) = \det(I - F_{1_*}) \cdots \det(I - F_{\ell_*}) = \det(I - F_*)$. $\square$

**Lemma 3.3** (*[9, Theorem 1.1]*). *Let $\pi, \pi' \subset \mathrm{Aff}(G)$ be two almost crystallographic groups. Then for any homomorphism $\theta : \pi \to \pi'$, there exists $(d, D) \in \mathrm{aff}(G)$ such that $\theta(\alpha) \cdot (d, D) = (d, D) \cdot \alpha$ for all $\alpha \in \pi$.*

Now we can prove our first main result which computes the Lefschetz number $L(f)$ and the Nielsen number $N(f)$ of any continuous map $f$ on an infra-nilmanifold $M$ in terms of the holonomy of the manifold.

**Theorem 3.4.** *Let $f : M \to M$ be any continuous map on an infra-nilmanifold $M$ with the holonomy group $\Psi$. Then*

$$L(f) = \frac{1}{|\Psi|} \sum_{A \in \Psi} \frac{\det(A_* - f_*)}{\det A_*},$$

$$N(f) = \frac{1}{|\Psi|} \sum_{A \in \Psi} |\det(A_* - f_*)|.$$

*In particular, $|L(f)| \le N(f)$.*

**Proof.** Let $\Lambda$ be a fully invariant subgroup of $\pi$ as in Lemma 3.1. Let $N = \Lambda \setminus G$ be the nilmanifold. Then, since $f_* : \pi \to \pi$ induces $f_* : \Lambda \to \Lambda$, there is a continuous map $\hat{f} : N \to N$ which makes the following diagram

$$\begin{array}{ccc} N & \xrightarrow{\hat{f}} & N \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{f} & M \end{array}$$

(where $p$ is the covering projection) commutative.

By Lemma 3.3, there exists $(d, D) \in G \rtimes \mathrm{Endo}(G)$ such that

$$f_*(\alpha) \cdot (d, D) = (d, D) \cdot \alpha$$

for all $\alpha \in \pi$. This implies that the map $(d, D) : G \to G$ induces a map $\Phi_{(d,D)} : M \to M$, and furthermore, $\Phi_{(d,D)}$ and $f$ induce exactly the same endomorphisms of $\pi_1(M) = \pi$. Since $M$ is a $K(\pi, 1)$-manifold, $\Phi_{(d,D)}$ and $f$ are homotopic. Thus, for our purposes, we can replace $f$ by $\Phi_{(d,D)}$ in the following discussions.

In particular, the above equality yields

$$f_*(\lambda) = (\mu(d) \circ D)(\lambda) \qquad \text{for all } \lambda \in \Lambda,$$

where $\mu(d)$ is conjugation by $d$.

Write $E = \mu(d) \circ D$. Then the map $E : G \to G$ restricts to a map

$$\phi_E : N \to N.$$

By the uniqueness of extension to $G$ (since $\Lambda$ is a lattice), the Lie algebra endomorphism $f_*$ induced by $f$ is exactly $\mu(d)_* \circ D_*$ so that $f_* = E : G \to G$ and $f_* = E_* : \mathfrak{G} \to \mathfrak{G}$. Furthermore, $\hat{f} \simeq \phi_E$. We shall replace $\hat{f}$ and $f$ in the diagram by $\phi_{(d,D)}$ and $\Phi_{(d,D)}$, respectively.

By [1],

$$L(\hat{f}) = \det(I - f_*) = \det(I - E_*)$$

$$N(\hat{f}) = |\det(I - f_*)| = |\det(I - E_*)|.$$

Let $\alpha \in \pi/\Lambda$. Choose a preimage $(a, A) \in \pi$ of $\alpha$ under the quotient map $\pi \to \pi/\Lambda$. Then $\alpha$ becomes a covering transformation of $\Lambda \setminus G$ via the following commutative diagram:

$$\begin{tikzcd} G \arrow[r, "{(a,A)}"] \arrow[d] & G \arrow[d] \\ \Lambda\backslash G \arrow[r, "\alpha"] & \Lambda\backslash G \end{tikzcd}$$

This means that $\alpha$ is the map induced by $(a, A) : G \to G$, i.e., $\alpha = \phi_{(a,A)}$. Moreover, $\alpha_* = (\mu(a) \circ A)_* = \mathrm{Ad}(a) \circ A_*$ on $\mathfrak{G}$. Consequently,

$$\begin{aligned} L(\alpha^{-1} \circ \hat{f}) &= \det(I - (\alpha_*^{-1} \circ E_*)) \\ &= \det(I - A_*^{-1}\mathrm{Ad}(a^{-1})E_*) \\ &= \det(I - \mathrm{Ad}(a^{-1})E_*A_*^{-1}) \\ &= \det(I - E_*A_*^{-1}) \qquad \text{by Lemma 3.2} \\ &= \frac{\det(A_* - f_*)}{\det A_*}. \end{aligned}$$

Recall that $\phi_{(a,A)}$ is an isomorphism of the lattice $\Gamma$. Therefore, $|\det A_*| = 1$, and we have $N(\alpha^{-1} \circ \hat{f}) = |L(\alpha^{-1} \circ \hat{f})| = \frac{|\det(A_* - f_*)|}{|\det A_*|} = |\det(A_* - f_*)|$. Therefore, by [5, Theorem III.2.12, (p. 52)] and by [7, Theorem 3.5], we have

$$\begin{aligned} L(f) &= \frac{1}{[\pi : \Lambda]} \sum_{\alpha \in \pi/\Lambda} L(\alpha^{-1} \circ \hat{f}) \\ &= \frac{1}{|\Psi|} \sum_{A \in \Psi} \frac{\det(A_* - f_*)}{\det A_*}, \\ N(f) &= \frac{1}{[\pi : \Lambda]} \sum_{\alpha \in \pi/\Lambda} |\det(A_* - f_*)| \\ &= \frac{1}{|\Psi|} \sum_{A \in \Psi} |\det(A_* - f_*)|. \end{aligned}$$

This finishes the proof of the Theorem. □

**Example 3.5.** Let $G$ be the 3-dimensional Heisenberg group. That is,

$$G = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$$

We denote this general element by $\{x, y, z\}$. The subgroup of $G$ consisting of all integral matrices forms a lattice $\Gamma$.

Let

$$a = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $A : G \to G$ be the automorphism of $G$ given by

$$A : \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -x & z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $A$ has period 2, and $(a, A)^2 = (a^2, I) = (\{0, 0, 1\}, I) \in G \rtimes \mathrm{Aut}(G)$, where $I$ is the identity automorphism of $G$. It is easy to check that the subgroup

$$\Pi = \langle \Gamma, (a, A) \rangle \subset G \rtimes \mathrm{Aut}(G)$$

generated by the lattice $\Gamma$ and the element $(a, A)$ is discrete and torsion free. Furthermore, $\Gamma$ is a normal subgroup of $\Pi$ of index 2. Thus, $\Pi$ is an almost Bieberbach group.

Let $K \in \mathrm{Aut}(G)$ be given by

$$K : \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -4x - y & z' \\ 0 & 1 & 6x + 2y \\ 0 & 0 & 1 \end{bmatrix},$$

where $z' = -2z - (12x^2 + 10xy + y^2)$. It is not difficult to check that $KA = AK$ and the conjugation by $(1, K) \in G \rtimes \mathrm{Aut}(G)$ maps $\Pi$ into $\Pi$ (and $\Gamma$ into $\Gamma$). Thus, the affine map $(1, K) : G \to G$ induces $\phi_K : \Gamma \setminus G \to \Gamma \setminus G$ and $\Phi_K : \Pi \setminus G \to \Pi \setminus G$ so that the following diagram is commutative:

$$\begin{array}{ccc} G & \xrightarrow{(1,K)} & G \\ \downarrow & & \downarrow \\ \Gamma\backslash G & \xrightarrow{\phi_K} & \Gamma\backslash G \\ \downarrow & & \downarrow \\ \Pi\backslash G & \xrightarrow{\Phi_K} & \Pi\backslash G \end{array}$$

Note that all the vertical maps are the natural covering maps. In particular, $\Gamma \setminus G \to \Pi \setminus G$ is a double covering by the holonomy group of $\Pi \setminus G$, which is $\Psi = \{I, A\} \cong \mathbb{Z}_2$.

To calculate $L(\Phi_K)$ and $N(\Phi_K)$, we take an ordered (linear) basis for the Lie algebra $\mathfrak{G}$ of $G$ as follows:

$$\mathbf{e}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{e}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

With respect to this basis, the differentials of $A$ and $K$ are

$$A_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad K_* = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & -1 \\ 0 & 6 & 2 \end{bmatrix}.$$

Therefore, the Nielsen number and the Lefschetz number of the map $\Phi_K : \Pi \setminus G \to \Pi \setminus G$ are:

$$\begin{aligned} N(\Phi_K) &= \frac{1}{|\Psi|}\sum_{\alpha\in\Psi} |\det(\alpha_* - K_*)| \\ &= \frac{1}{2}(|\det(I_* - K_*)| + |\det(A_* - K_*)|) \\ &= \frac{1}{2}(|3| + |-9|) \\ &= 6, \\ L(\Phi_K) &= \frac{1}{|\Psi|}\sum_{\alpha\in\Psi} \frac{\det(\alpha_* - K_*)}{\det(\alpha_*)} \\ &= \frac{1}{2}\left(\frac{3}{1} + \frac{-9}{1}\right) \\ &= -3. \end{aligned}$$

Hence $|L(\Phi_K)| < N(\Phi_K)$.

## 4. Sets of periods for expanding maps on infra-nilmanifolds

Let $M$ be a closed manifold. A smooth map $f : M \to M$ is called *an expanding map* if there exist constants $C > 0$ and $\mu > 1$ such that $\|Df^n(v)\| \geq C\mu^n\|v\|, \forall v \in T(M)$, for some Riemannian metric $\|\ \|$ on $M$. In [4], it was shown that any expanding map on a closed manifold is topologically conjugate to an expanding map on an infra-nilmanifold. Let $M = \pi \setminus G$ be an infra-nilmanifold and let $f : M \to M$ be an expanding map on $M$. Then by [15], the induced homomorphism $f_* : \pi \to \pi$ on the almost Bieberbach group $\pi$ is injective. By [10], there is $(k, K) \in \mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$ such that for all $\alpha \in \pi$,

$$f_*(\alpha) = (k, K) \cdot \alpha \cdot (k, K)^{-1}.$$

The affine map $(k, K) : G \to G$ induces a differentiable map $\Phi_{(k,K)} : M \to M$, which is expanding. That is, all of the eigenvalues of the induced automorphism $K_*$ on the Lie algebra $\mathfrak{G}$ by the Lie group automorphism $K$ on $G$ are of modulus $> 1$. Two expanding maps $f$ and $\Phi_{(k,K)}$ on $M$ induce the same homomorphism on the deck transformation group $\pi$ of the universal covering space $G$ of the manifold $M$ and hence by [15], they are topologically conjugate, i.e., there exists a homeomorphism $h$ of $M$ such that

$$f = h^{-1} \circ \Phi_{(k,K)} \circ h.$$

For this reason, from now on, we may assume that there is $(k, K) \in \mathrm{Aff}(G)$ so that $(k, K)$ conjugates $\pi$ into itself, and the map $\Phi_{(k,K)} : \pi \setminus G \to \pi \setminus G$ induced by the affine map $(k, K) : G \to G$ is our expanding map.

Let $\Gamma = G \cap \pi$ and $N = \Gamma \setminus G$. Then $\Psi = \Gamma \setminus \pi$ is the holonomy group of $M$. For $K \in \mathrm{Aut}(G)$, $K_* \in \mathrm{Aut}(\mathfrak{G})$ denotes the differential of $K$. Since $(k, K)\Gamma(k, K)^{-1} \subset \Gamma$, $(k, K)$

induces a map $\phi_{(k,K)} : N \to N$. Moreover the following diagram is commutative

$$\begin{array}{ccc} G & \xrightarrow{(k,K)} & G \\ {\scriptstyle \Gamma\backslash}\big\downarrow{\scriptstyle p} & & {\scriptstyle \Gamma\backslash}\big\downarrow{\scriptstyle p} \\ N & \xrightarrow{\phi_{(k,K)}} & N \\ {\scriptstyle \Psi\backslash}\big\downarrow{\scriptstyle q} & & {\scriptstyle \Psi\backslash}\big\downarrow{\scriptstyle q} \\ M & \xrightarrow{\Phi_{(k,K)}} & M. \end{array}$$

It is known in [15] that any expanding map $f : M \to M$ is a finite covering map, and for all $k \geq 1$, $f^k$ is also an expanding map. Moreover, the set of fixed points, $\mathrm{Fix}(f)$, is non-empty and finite, and the set of periodic points, $\bigcup_{k\geq 1} \mathrm{Fix}(f^k)$, is countable and dense in $M$. By Proposition 1 of [8], the path components in $\mathrm{Fix}(f)$ are the fixed point classes in the infra-nilmanifold. Since the isolated fixed point classes are essential with local fixed point index $\pm 1$, we have $|\mathrm{Fix}(f)| = N(f)$.

**Lemma 4.1.** *Let $f = \Phi_{(k,K)}$ be an expanding map on the infra-nilmanifold $M$ with holonomy group $\Psi$. Then*

$$|\mathrm{Fix}(f)| = \frac{1}{|\Psi|} \sum_{A\in\Psi} |\det(A_* - K_*)|.$$

**Proof.** Note that $\hat{f} = \phi_{(k,K)} \simeq \phi_{\mu(k)\circ K}$ and $f_* = \mathrm{Ad}(k)K_* : \mathfrak{G} \to \mathfrak{G}$. Thus by Lemma 3.2, $\det(A_* - f_*) = \det(A_* - \mathrm{Ad}(k)K_*) = \det(A_* - K_*)$. Hence $|\mathrm{Fix}(f)| = N(f) = \frac{1}{|\Psi|} \sum_{A\in\Psi} |\det(A_* - K_*)|$. $\square$

**Lemma 4.2.** *Let $\Gamma$ be a lattice of $G$, and let $K \in \mathrm{Aut}(G)$ such that $K(\Gamma) \subset \Gamma$. For some choice of basis in $\mathfrak{G}$, $K_* \in \mathrm{GL}(n, \mathbb{Q})$.*

**Proof.** This is a consequence of Mal'cev [11]. Since $\Gamma$ is a lattice in the $n$-dimensional connected and simply connected nilpotent Lie group $G$ with compact quotient $N = \Gamma \setminus G$, $\Gamma$ is a torsion-free and finitely generated nilpotent group and so has a central series

$$\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n \supset \Gamma_{n+1} = 1$$

with $\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}$ for each $i = 1, 2, \ldots, n$. Choose a set of generators for $\Gamma$:

$$a_1, a_2, a_3, \ldots, a_n$$

in such a way that $\Gamma_i$ is the group generated by $\{a_i\} \cup \Gamma_{i+1}$, for each $i = 1, 2, \ldots, n$. We refer to $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ as a *canonical basis* of $\Gamma$. Given a canonical basis $\mathbf{a}$ of $\Gamma$, any element $\gamma \in \Gamma$ can be uniquely expressed as a product

$$\gamma = a_1^{z_1} a_2^{z_2} \cdots a_n^{z_n}, \qquad \text{with } \vec{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{Z}^n.$$

Now we regard $G$ as the Mal'cev completion of $\Gamma$:

$$G = \{a_1^{z_1} a_2^{z_2} \cdots a_n^{z_n} \mid \vec{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n\}.$$

Given $\vec{z} \in \mathbb{Z}^n$, we use $\gamma(\vec{z})$ to denote the element of $\Gamma$ whose canonical coordinate is $\vec{z}$. Thus, we have an identification $\gamma : \mathbb{Z}^n \to \Gamma$ sending $\vec{z}$ to $\gamma(\vec{z})$. Among interesting properties of this identification, we recall the following ([3, Theorem 2.1.(3)]): for any homomorphism

$\varphi : \Gamma \to \Gamma$, there exists a polynomial function (with rational coefficients) $\psi_\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $\varphi(\gamma(\vec{z})) = \gamma(\psi_\varphi(\vec{z}))$ for all $\vec{z} \in \mathbb{Z}^n$. Moreover, any homomorphism of $\Gamma$ extends to a homomorphism of $G$ by using the same polynomial. Thus the left commutative diagram extends to the right commutative diagram:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\varphi} & \Gamma \\ {\scriptstyle\gamma}\uparrow & & \uparrow{\scriptstyle\gamma} \\ \mathbb{Z}^n & \xrightarrow{\psi_\varphi} & \mathbb{Z}^n \end{array} \Longrightarrow \begin{array}{ccc} G & \xrightarrow{\varphi} & G \\ {\scriptstyle\gamma}\uparrow & & \uparrow{\scriptstyle\gamma} \\ \mathbb{R}^n & \xrightarrow{\psi_\varphi} & \mathbb{R}^n \end{array}$$

Now we consider another identification of $G$ with $\mathbb{R}^n$ using the associated Lie algebra $\mathfrak{G}$. Fix an ordered basis for the vector space $\mathfrak{G}$ and identify $\mathfrak{G}$ with $\mathbb{R}^n$ by mapping each element to its coordinate. This gives the second identification of $G$ with $\mathbb{R}^n$:

$$\log : G \xrightarrow{\log} \mathfrak{G} \cong \mathbb{R}^n.$$

Then by Theorem 2.2 of [3], there exist polynomial maps with rational coefficients $p_1, p_2 : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\log \circ \gamma(\vec{q}) = p_1(\vec{q}) \qquad \text{and} \qquad \log \circ \gamma \circ p_2(\vec{q}) = \vec{q}, \qquad \text{all } \vec{q} \in \mathbb{R}^n.$$

Now, for the given $K \in \mathrm{Aut}(G)$, since $K(\Gamma) \subset \Gamma$, it induces a polynomial function $\psi_K$ of rational coefficients. We have the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{R}^n \cong \mathfrak{G} & \xleftarrow{\log} & G & \xleftarrow{\gamma} & \mathbb{R}^n \\ \downarrow{\scriptstyle K_*} & & \downarrow{\scriptstyle K} & & \downarrow{\scriptstyle \psi_K} \\ \mathbb{R}^n \cong \mathfrak{G} & \xleftarrow{\log} & G & \xleftarrow{\gamma} & \mathbb{R}^n \end{array}$$

Chasing the diagram yields that $K_*$ is a polynomial map with rational coefficients: For,

$$\begin{aligned} K_* &= \log \circ K \circ \exp = \log \circ K \circ \exp \circ (\log \circ \gamma \circ p_2) \\ &= \log \circ K \circ \gamma \circ p_2 = \log \circ \gamma \circ \psi_K \circ p_2 = p_1 \circ \psi_K \circ p_2. \end{aligned}$$

Hence for some basis for $\mathfrak{G}$, $K_*$ is a matrix with rational entries. □

**Definition 4.3.** The *set of periods* of $f : M \to M$ is

$$\mathcal{P}(f) = \left\{ m \in \mathbb{N} \mid \mathrm{Fix}(f^m) \setminus \bigcup_{i=1}^{m-1} \mathrm{Fix}(f^i) \neq \emptyset \right\}.$$

We know that the fixed point set $\mathrm{Fix}(f^m)$ is a finite set for all $m \geq 1$ and the set of periods $\mathcal{P}(f)$ is infinite. However, some periods may be missing. For example, the following is known in [16]: consider the expanding map $\Phi_{-2I}$ on the torus $T^d$ induced by the linear map $-2I : \mathbb{R}^d \to \mathbb{R}^d$. Then $\mathrm{Fix}(\Phi_{-2I}) \subset \mathrm{Fix}(\Phi_{-2I}^2)$ and, by Lemma 4.1,

$$\begin{aligned} |\mathrm{Fix}(\Phi_{-2I}^2) \setminus \mathrm{Fix}(\Phi_{-2I})| &= |\mathrm{Fix}(\Phi_{-2I}^2)| - |\mathrm{Fix}(\Phi_{-2I})| \\ &= |\det(I - (-2I)^2)| - |\det(I - (-2I))| \\ &= |(-3)^d| - 3^d \\ &= 0. \end{aligned}$$

That is, 2 is not a period of $\Phi_{-2I}$. Thus $2 \notin \mathcal{P}(\Phi_{-2I})$.

Jiang and Llibre showed in [6] that there is a positive integer $m_0$ such that for any integer $m \geq m_0$ and for any expanding map $f$ of the $d$-torus $T^d$, there exists a periodic point of $f$ whose least period is exactly $m$. In [16], Tauraso extended this result to each flat manifold. In the following theorem we prove that the above "uniform cofiniteness property" of sets of periods for expanding maps is true not only for flat manifolds but also infra-nilmanifolds. Since any closed manifold which admits an expanding map is homeomorphic to an infra-nilmanifold, our result is the strongest generalization.

Recall the following Liouville's inequality (see the Introduction of [13]).

**Lemma 4.4.** *Let $\alpha$ be an algebraic number of degree $d$, with minimal polynomial*

$$a_0X^d + a_1X^{d-1} + \cdots + a_d = a_0(X - \alpha_1)\cdots(X - \alpha_d),$$

*where $a_0, \ldots, a_d$ are relatively prime integers and $a_0 > 0$. We denote by $M(\alpha)$ Mahler's measure of $\alpha$: $M(\alpha) = a_0 \prod_{i=1}^{d} \max\{1, |\alpha_i|\}$. If $\alpha \neq 1$, then*

$$|\alpha - 1| \geq \frac{1}{2^{d-1}M(\alpha)}.$$

Recall a result of M. Mignotte (Corollary A.3 of [6]).

**Lemma 4.5.** *Let $\alpha$ be a nonzero algebraic number of degree $d$. Then $|\alpha|$ is an algebraic number with degree at most $d(d-1)$ and with Mahler's measure $M(|\alpha|)$ at most $M(\alpha)^{d-1}$.*

Let $\alpha$ be an algebraic number. If $|\alpha| \neq 1$, then Lemma 4.4 together with Lemma 4.5, gives

$$||\alpha| - 1| \geq \frac{1}{2^{d(d-1)-1}M(\alpha)^{d-1}} \geq \frac{1}{2^{d^2}M(\alpha)^d}. \tag{*}$$

This inequality will play an essential role in proving the next theorem.

The following theorem generalizes the main result, Theorem 3.3, of [16]. The crucial point is our Lemmas 4.1 and 4.2. Then we just follow the argument of [16, Theorem 3.3].

**Theorem 4.6.** *Let $n$ be a positive integer. Then there exists a positive integer $m_0$, which depends only on $n$, such that for any integer $m \geq m_0$, for any $n$-dimensional infra-nilmanifold $M$, and for any expanding map $f$ on $M$, there exists a periodic point of $f$ whose least period is exactly $m$.*

**Proof.** Let $M$ be an $n$-dimensional infra-nilmanifold and $f$ an expanding map on $M$. Then we may assume that $M = \pi \setminus G$ with holonomy group $\Psi$ and $f = \Phi_{(k,K)}$. Let $\alpha_1, \ldots, \alpha_n$ be the eigenvalues of $K_*$ and let $\rho'(K) = \max\{|\alpha_1|, \ldots, |\alpha_n|\}$. Since $f$ is an expanding map, all $|\alpha_i| > 1$.

By Lemma 4.2, the characteristic polynomial of $K$ has rational coefficients so that all the eigenvalues satisfy the conditions of Lemmas 4.4 and 4.5.

First we observe that, for any $A \in \Psi$ and for any $j, \ell \geq 1$,

$$(K^\ell A^{-1})^j = (K^\ell A^{-1}K^{-\ell})(K^{2\ell}A^{-1}K^{-2\ell})\cdots(K^{j\ell}A^{-1}K^{-j\ell})K^{j\ell}.$$

Since $(k, K)\pi(k, K)^{-1} \subset \pi$, we have $K\Psi K^{-1} \subset \Psi$, and $\Psi$ is a finite group of order $|\Psi|$. Thus for any $\ell \geq 1$, there is an integer $1 \leq j_0 \leq |\Psi|$ such that $K^{j_0\ell}A^{-1}K^{-j_0\ell} = A^{-1}$. Put

$$B = (K^\ell A^{-1}K^{-\ell})(K^{2\ell}A^{-1}K^{-2\ell})\cdots(K^{j_0\ell}A^{-1}K^{-j_0\ell}).$$

Then $B \in \Psi$ and so $B^{|\Psi|} = I$. Since $K^{j_0\ell}$ commutes with each factor of $B$ in the above expression, $B$ and $K^{j_0\ell}$ commute. Hence

$$(K^\ell A^{-1})^{j_0|\Psi|} = B^{|\Psi|} K^{j_0\ell|\Psi|} = (K^\ell)^{j_0|\Psi|}.$$

This means that the eigenvalues of $K_*^\ell {A_*}^{-1}$ and $K_*^\ell$ are the same in absolute value: $|\alpha_1|^\ell, \ldots, |\alpha_n|^\ell$. Hence we have

$$\prod_{i=1}^n (|\alpha_i|^\ell - 1) \le |\det(K_*^\ell {A_*}^{-1} - I)| \le \prod_{i=1}^n (|\alpha_i|^\ell + 1).$$

By Lemma 4.1,

$$|\mathrm{Fix}(f^\ell)| = \frac{1}{|\Psi|} \sum_{A \in \Psi} |\det(K_*^\ell {A_*}^{-1} - I)|.$$

Therefore, for $m, \ell \ge 1$,

$$\frac{|\mathrm{Fix}(f^m)|}{|\mathrm{Fix}(f^\ell)|} = \frac{\sum_{A\in\Psi} |\det(K_*^m {A_*}^{-1} - I)|}{\sum_{A\in\Psi} |\det(K_*^\ell {A_*}^{-1} - I)|} \ge \prod_{i=1}^n \frac{|\alpha_i|^m - 1}{|\alpha_i|^\ell + 1}.$$

Now the eigenvalues $\alpha_1, \ldots, \alpha_n$ are algebraic numbers greater than 1 in absolute value. The minimal polynomial of each $\alpha_i$ has degree $d_i \le n$ with leading coefficient say $a_i$. Let $a_0 = \max\{a_1, \ldots, a_n\}$ and let $\rho(K) = a_0 \cdot \rho'(K)$. Then $1 < M(\alpha_i) \le \rho(K)^n$. Hence by (*),

$$|\alpha_i| - 1 \ge \frac{1}{2^{d_i^2} M(\alpha_i)^{d_i}} \ge \frac{1}{2^{n^2} \rho(K)^{n^2}}.$$

Let $1 \le \ell \le \frac{m}{2}$. Then

$$\frac{|\alpha_i|^m - 1}{|\alpha_i|^\ell + 1} \ge \frac{|\alpha_i|^\ell}{|\alpha_i|^\ell + 1}(|\alpha_i|^{m-\ell} - 1) \ge \frac{|\alpha_i| - 1}{2} \ge \frac{1}{2^{n^2+1}\rho(K)^{n^2}},$$

and hence

$$\frac{|\mathrm{Fix}(f^m)|}{|\mathrm{Fix}(f^\ell)|} \ge \left(\frac{1}{2^{n^2+1}\rho(K)^{n^2}}\right)^{n-1} \cdot \frac{\rho(K)^m - 1}{\rho(K)^\ell + 1} \ge \frac{\rho(K)^{\frac{m}{2}} - 1}{2^{(n^2+1)(n-1)}\rho(K)^{n^2(n-1)}}.$$

Since $\rho(K) > 1$, there is $m_0 \ge 2n^2(n-1)$ such that for all $m \ge m_0$,

$$g(m) = [\rho(K)^{\frac{m}{2}} - 1] - (2^{(n^2+1)(n-1)}\rho(K)^{n^2(n-1)})\frac{m}{2} > 0.$$

Hence there is $m_0 = m_0(n)$ such that for all $m \ge m_0$ and $1 \le \ell \le \frac{m}{2}$,

$$\frac{|\mathrm{Fix}(f^m)|}{|\mathrm{Fix}(f^\ell)|} > \frac{m}{2}.$$

Let $x \in \mathrm{Fix}(f^m)$. Then it has a least period $\ell$ with $1 \le \ell \le m$ and $\ell$ divides $m$. Therefore, the set of periodic points of least period $m$ for $f$ is

$$\mathrm{Fix}(f^m) \setminus \bigcup_{i=1}^{m-1} \mathrm{Fix}(f^i) = \mathrm{Fix}(f^m) \setminus \bigcup_{\ell | m, \ell < m} \mathrm{Fix}(f^\ell).$$

For $m \geq m_0$, the number of periodic points of least period $m$ for $f$ is

$$\begin{aligned}
&\left|\mathrm{Fix}(f^m) \setminus \bigcup_{\ell | m, \ell < m} \mathrm{Fix}(f^\ell)\right| \\
&\quad > |\mathrm{Fix}(f^m)| - \sum_{\ell | m, \ell < m} |\mathrm{Fix}(f^\ell)| \\
&\quad > |\mathrm{Fix}(f^m)| \left(1 - \sum_{1 \leq \ell \leq \frac{m}{2}} \frac{2}{m}\right) > 0.
\end{aligned}$$

That is, for any $m \geq m_0$, there exists a periodic point of $f$ whose least period is exactly $m$. □

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